# Analysis of Control-Output Interactions in Dynamic Systems

A general theory for examining the dynamic as well as static interactions among inputs and outputs in a multivariable control system has been developed. Both state space and frequency domain approaches are used, and at steady state the interaction equations assume the same form as the relative gain array developed by Bristol. Implications of selecting input-output pairings with regard to steady state vs. dynamic consideration are discussed. The application of the method of analysis to a distillation column control system is presented.

L. S. TUNG
and
T. F. EDGAR
Department of Chemical Engineering
University of Texas
Austin, Texas 78712

Virtually all chemical processes are multivariable in nature, i.e., they are characterized by multiple inputs (controls) and multiple outputs. Industrial control systems generally are designed by assuming that the multivariable control problem can be decomposed into a series of single input-single output problems. Often the individual control-output pairs are selected intuitively, although the so-called relative gain array (Bristol, 1966; Shinskey, 1967) has been used successfully in industrial applications to determine the best pairings.

The relative gain array (RGA) is based on a steady state analysis, although Bristol (1966, 1978) has described ways in which dynamic properties of the closed loop system can be inferred from the RGA. However, this qualitative approach is sometimes not sufficient for control loop design. There is a need to place the interaction analysis for controls and outputs on a more quantitative basis, which accounts directly for dynamic properties of the system. In this paper we present a dynamic RGA for square linear systems. The proposed approach is completely general and the elements in the array are independent of the initial conditions. The Bristol array is shown to be a special case of the dynamic RGA. McAvoy (1978) and Gagnepain and Seborg (1980) have recently reported alternative methods for analyzing dynamic interactions.

We present the theory of the dynamic interaction index both in the time and frequency domains. The results for these models illustrate that the dynamic RGA can be applied to any system order, and clarifies those cases where the static RGA gives misleading or incomplete predictions about closed loop behavior for a given pairing.

## THEORETICAL DEVELOPMENT

The basic strategy of the static relative gain array (Bristol, 1966) is to choose a control loop in which the manipulated variable ($u_j$) and the controlled variable ($y_i$) are most sensitive to each other and hence less sensitive to other input-output pairs. Because of interaction effects, one must consider the so-called open loop sensitivity:

$$\left(\frac{\partial y_i}{\partial u_j}\right)_{uk,k\neq j}$$

as well as the closed loop sensitivity:

$$\left(\frac{\partial y_i}{\partial u_j}\right)_{yk,k\neq i}$$

where $u_{k,k\neq j}$ indicates all controllers except $u_j$ are held constant. A measure of relative sensitivity is given by the relative gain array, whose elements are defined as follows:

$$\alpha_{ij} = \frac{\left(\frac{\partial y_i}{\partial u_j}\right)_{uk,k\neq j}}{\left(\frac{\partial y_i}{\partial u_j}\right)_{yk,k\neq i}} \tag{1}$$

Bristol has shown that the proper input-output ($y_i - u_j$) pair for single loop control is the one having the largest positive $\alpha_{ij}$ value. However, this relative gain analysis is a steady state analysis and does not explicitly include dynamic effects. A method for dynamic as well as steady state analysis will now be developed. We shall present results based on both time domain (state-space) and frequency domain analyses. In addition, we show how the static relative gain array naturally arises out of analysis of the time and frequency domain responses.

Let a linear dynamic process be described by:

$$\dot{x} = Ax + Bu \tag{2}$$

$$y = Cx \tag{3}$$

where

$x = n \times 1$ State Vector

$y = m \times 1$ Output Vector

$u = m \times 1$ Control Vector

The system is assumed to be controllable and observable.

Consider a change in set point from zero to $y^o$, and let the required control change necessary to bring about this set point change be $u = u^o$. At steady state,

$$0 = Ax^o + Bu^o \tag{4}$$

Assuming the system is stable, then the above equation can be solved for $x^o$,

$$x^o = (-A)^{-1}Bu^o \tag{5}$$

but

$$\begin{aligned} y^o &= Cx^o \\ &= C(-A)^{-1}Bu^o \end{aligned} \tag{6}$$

therefore, for $y$ and $u$ of the same dimension,

$$u^o = [C(-A)^{-1}B]^{-1}y^o \tag{7}$$

With zero initial conditions, the output response is, in the $s$-domain,

$$\begin{aligned} y(s) &= Cx(s) \\ &= C(sI - A)^{-1}Bu^o/s \\ &= C(sI - A)^{-1}B \cdot [C(-A)^{-1}B]^{-1}y^o/s \\ &= \phi(s) \cdot [\phi(o)]^{-1}y^o/s \end{aligned} \tag{8}$$

where $\phi(s)$ is the process transfer function matrix. Equation 8 can be written in more detail,

$$\begin{bmatrix} y_1(s) \\ y_2(s) \\ \vdots \\ y_m(s) \end{bmatrix} = \begin{bmatrix} \phi_{1,1}(s) & \phi_{1,2}(s) & \cdots & \phi_{1,m}(s) \\ \phi_{2,1}(s) & \phi_{2,2}(s) & \cdots & \phi_{2,m}(s) \\ \vdots & \vdots & & \vdots \\ \phi_{m,1}(s) & \phi_{m,2}(s) & \cdots & \phi_{m,m}(s) \end{bmatrix}$$
$$\begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \cdots & \Gamma_{1,m} \\ \Gamma_{2,1} & \Gamma_{2,2} & \cdots & \Gamma_{2,m} \\ \vdots & \vdots & & \vdots \\ \Gamma_{m,1} & \Gamma_{m,2} & \cdots & \Gamma_{m,m} \end{bmatrix} \cdot \begin{bmatrix} y_1^o \\ y_2^o \\ \vdots \\ y_m^o \end{bmatrix} \cdot \frac{1}{s} \tag{9}$$

0001-1541-81-1945-0691-$2.00. 

that is,

$$y_i(s) = \sum_{j=1}^{m} \left( \sum_{k=1}^{m} \phi_{i,k}(s) \Gamma_{k,j} \right) \frac{y_j^o}{s}, \ i = 1,2 \ldots, m \quad (10)$$

where $\phi_{i,j}(s)$ and $\Gamma_{k,j}$ are elements of the $C(sI - A)^{-1}B$ matrix and the $[C(-A)^{-1}B]^{-1}$ matrix $(\phi^{-1}(o))$, respectively.

Now consider a step change in $y_i^o$ only. The response of $y_i$ is simply:

$$y_i(s) = \left( \sum_{k=1}^{m} \phi_{i,k}(s) \Gamma_{k,i} \right) \frac{y_i^o}{s} \quad (11)$$

Note that the $k$th term in the summation results from the $k$th controller. The above equation indicates that if $y_i$ is to be controlled by controller $u_r$, the term $\phi_{i,r}\Gamma_{r,i}/s$ should be the dominant term. A dynamic relative gain matrix can be formed from Eq. 11. With each $i$-th row of the matrix formed from the $i$-th component of Eq. 11, the following dynamic relative gain matrix can be defined.

$$\begin{array}{c|cccc} & u_1 & u_2 & \ldots & u_m \\ \hline y_1 & \alpha_{1,1}(s) & \alpha_{1,2}(s) & \ldots & \alpha_{1,m}(s) \\ y_2 & \alpha_{2,1}(s) & \alpha_{2,2}(s) & \ldots & \alpha_{2,m}(s) \\ \vdots & \vdots & \vdots & & \vdots \\ y_m & \alpha_{m,1}(s) & \alpha_{m,2}(s) & \ldots & \alpha_{m,m}(s) \end{array} \quad (12)$$

where

$$\alpha_{i,j}(s) = \phi_{i,j}(s)\Gamma_{j,i}/s, \quad i = 1, 2, \ldots, m \qquad j = 1, 2, \ldots, m \quad (13)$$

Having presented a frequency domain version of the dynamic RGA the time domain interpretation will next be developed. Equation 11 can be inverted to the time domain as follows:

$$y_i(t) = y_i^o \sum_{k=1}^{m} (\hat{a}_o^{i,k} + \hat{a}_1^{i,k} e^{\lambda_1 t} + \ldots + \hat{a}_n^{i,k} e^{\lambda_n t}) \Gamma_{k,i} \quad (14)$$

Let $a_j^{i,k} = \hat{a}_j^{i,k}\Gamma_{k,i}$ for $j = 0, 1, 2, \ldots, n$; then the above equation can be written more simply as:

$$\frac{y_i(t)}{y_i^o} = \left( a_o^{i,1} + \sum_{j=1}^{m} a_j^{i,1} e^{\lambda_j t} \right) + \left( a_o^{i,2} + \sum_{j=1}^{n} a_j^{i,2} e^{\lambda_j t} \right) + \ldots + \left( a_o^{i,m} + \sum_{j=1}^{n} a_j^{i,m} e^{\lambda_j t} \right) \quad (15)$$

An important implication of this equation is that the right hand side is independent of the initial conditions and is dimensionless.

Now suppose $y_i(t)$ is controlled by controller $u_r$. Observe that in the above equation, the term

$$\left( a_o^{i,1} + \sum_{j=1}^{m} a_j^{i,1} e^{\lambda_j t} \right)$$

arises because of controller $u_1$; the term

$$\left( a_o^{i,2} + \sum_{j=1}^{m} a_j^{i,2} e^{\lambda_j t} \right)$$

arises because of controller $u_2, \ldots$, etc. Multivariable interaction for each input-output pair can be calculated as a function of time using the above expressions. A modal analysis of Eq. 15 may be beneficial in identifying dominant interactions, as discussed by Tung and Edgar (1977).

At steady state, Eq. 15 becomes

$$\frac{y_i(t \to \infty)}{y_i^o} = a_o^{i,1} + a_o^{i,2} + \ldots + a_o^{i,m} = 1 \quad (16)$$

Note that

$$a_o^{i,k} = \phi_{i,k}(s = 0)\Gamma_{k,i} = \left.\frac{\partial y_i}{\partial u_k}\right|_u \left.\frac{\partial u_k}{\partial y_i}\right|_y \quad (17)$$

hence the terms in Eq. 16 simply appear in the $i$-th row of the relative gain matrix

$$\begin{array}{c|ccc} & u_1 & \ldots & u_m \\ \hline y_1 & \left(\left.\frac{\partial y_1}{\partial u_1}\right|_u\right)\Big/\left(\left.\frac{\partial y_1}{\partial u_1}\right|_y\right) & \ldots & \left(\left.\frac{\partial y_1}{\partial u_m}\right|_u\right)\Big/\left(\left.\frac{\partial y_1}{\partial u_m}\right|_y\right) \\ \vdots & \vdots & & \vdots \\ y_m & \left(\left.\frac{\partial y_m}{\partial u_1}\right|_u\right)\Big/\left(\left.\frac{\partial y_m}{\partial u_1}\right|_y\right) & \ldots & \left(\left.\frac{\partial y_m}{\partial u_m}\right|_u\right)\Big/\left(\left.\frac{\partial y_m}{\partial u_m}\right|_y\right) \end{array}$$

Therefore, we see how the notion of the static relative gain array can be derived from a detailed dynamic analysis. It should be mentioned that Bristol's RGA was originally derived intuitively without a rigorous analysis such as this.

## APPLICATIONS OF THE DYNAMIC RGA

*Example 1.* Let us consider a two by two process transfer function, $y(s) = \phi(s)u(s)$. From Eqs. 12 and 13:

$$\alpha_{i,j}(s) = \phi_{i,j}(s)\Gamma_{j,i}/s, \quad i = 1, 2 \qquad j = 1, 2 \quad (18)$$

$$[\Gamma_{i,j}] = [\phi(s = 0)]^{-1}$$

thus,

$$\begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} \\ \Gamma_{2,1} & \Gamma_{2,2} \end{bmatrix} = \begin{bmatrix} \phi_{1,1}^o & \phi_{1,2}^o \\ \phi_{2,1}^o & \phi_{2,2}^o \end{bmatrix}^{-1} = \frac{1}{\phi_{1,1}^o\phi_{2,2}^o - \phi_{1,2}^o\phi_{2,1}^o} \begin{bmatrix} \phi_{2,2}^o & -\phi_{1,2}^o \\ -\phi_{2,1}^o & \phi_{1,1}^o \end{bmatrix} \quad (19)$$

where

$$\phi_{i,j}^o \equiv \phi_{i,j}(s = 0) \quad (20)$$

Substituting $\Gamma_{i,j}$ values from Eq. 19 into Eq. 18, the following dynamic RGA is obtained for the two by two system:

$$\begin{aligned} \alpha_{1,1}(s) &= \frac{\phi_{1,1}(s)\phi_{2,2}^o}{\phi_{1,1}^o\phi_{2,2}^o - \phi_{1,2}^o\phi_{2,1}^o} \frac{1}{s} \\ \alpha_{1,2}(s) &= \frac{-\phi_{1,2}(s)\phi_{2,1}^o}{\phi_{1,1}^o\phi_{2,2}^o - \phi_{1,2}^o\phi_{2,1}^o} \frac{1}{s} \\ \alpha_{2,1}(s) &= \frac{-\phi_{2,1}(s)\phi_{1,2}^o}{\phi_{1,1}^o\phi_{2,2}^o - \phi_{1,2}^o\phi_{2,1}^o} \frac{1}{s} \\ \alpha_{2,2}(s) &= \frac{\phi_{2,2}(s)\phi_{1,1}^o}{\phi_{1,1}^o\phi_{2,2}^o - \phi_{1,2}^o\phi_{2,1}^o} \frac{1}{s} \end{aligned} \quad (21)$$

*Example 2.* This example considers an eighteen state distillation column model which is a linearized version of the nonlinear, material-energy-momentum balance model presented by Tung and Edgar (1979). The column has eight sieve trays and is used to separate methanol (MeOH) and water. The linearized model is described by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x &= [x_1, x_2, \ldots, x_{10}, L_2, L_3, \ldots, L_9]^T \\ u &= [u_1, u_2]^T \\ [y_1, y_2] &= [x_1, x_{10}] \end{aligned} \quad (22)$$

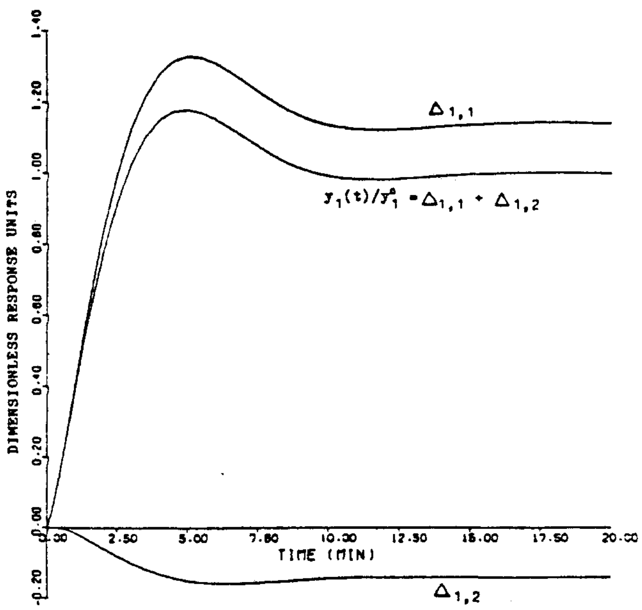


Figure 1. Interaction terms for $y_1$ of the distillation column.

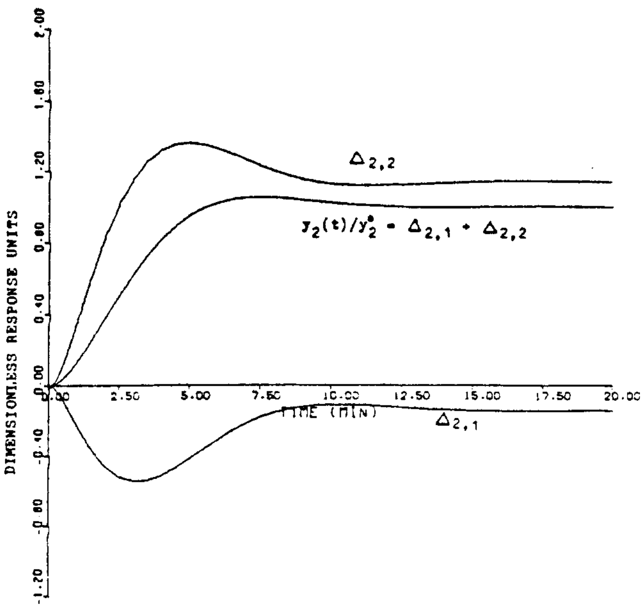


Figure 2. Interaction terms for $y_2$ of the distillation column.

where

- $x_1$ = MeOH concentration deviation of bottom product, mole fraction.
- $x_{10}$ = MeOH concentration deviation of top product, mole fraction.
- $u_1$ = reboiler steam pressure deviation
- $u_2$ = reflux molar flow rate deviation
- $L_i$ = deviation of liquid molar flow rate from tray $i$

Numerical values of the $A$ and $B$ matrices and other pertinent information have been given by Tung and Edgar (1979).

The system eigenvalues are $-0.3333 \pm 0.4851j$, $-0.5289$, $-2.169 \pm 0.3339j$, $-4.376 \pm 0.3218j$, $-5.251$, $-5.515$, $-7.0$, $-7.488 \pm 1.685j$, $-8.580$, $-9.563$, $-93.18 \pm 2.827j$, $-99.11 \pm 1.702j$. From Eq. 15

$$y_1(t)/y_1^o = \left(a_o^{1,1} + \sum_{k=1}^{18} a_k^{1,1} e^{\lambda_k t}\right) + \left(a_o^{1,2} + \sum_{k=1}^{18} a_k^{1,2} e^{\lambda_k t}\right) = \Delta_{1,1} + \Delta_{1,2}$$

$$y_2(t)/y_2^o = \left(a_o^{2,1} + \sum_{k=1}^{18} a_k^{2,1} e^{\lambda_k t}\right) + \left(a_o^{2,2} + \sum_{k=1}^{18} a_k^{2,2} e^{\lambda_k t}\right) = \Delta_{2,1} + \Delta_{2,2} \quad (23)$$

For $y_1$,

$$\Delta_{1,1} = 1.142 - 1.234e^{-0.3333t}\cos(0.4851t) + 5.123 \times 10^{-2}e^{-0.3333t}\sin(0.4815t) + 7.369 \times 10^{-2}e^{-0.5289t} + \ldots \quad (24)$$

$$\Delta_{1,2} = -0.142 + 0.1417e^{-0.3333t}\cos(0.4851t) + 9.458 \times 10^{-2}e^{-0.3333t}\sin(0.4815t) - 5.458 \times 10^{-4}e^{-0.5289t} + \ldots$$

Obviously, $y_1$ is dominated strongly by controller $u_1$, both during the transient phase and at steady state.

For $y_2$,

$$\Delta_{2,1} = -0.142 + 0.8726e^{-0.3333t}\cos(0.4851t) - 0.8784e^{-0.3333t}\sin(0.4815t) - 0.5588e^{-0.5289t} + \ldots \quad (25)$$

$$\Delta_{2,2} = 1.142 - 1.360e^{-0.3333t}\cos(0.4851t) + 0.2176e^{-0.3333t}\sin(0.4815t) - 3.338 \times 10^{-2}e^{-0.5289t} + \ldots$$

Although at steady state $y_2$ is strongly dominated by controller $u_2$, during the transient phase there is considerable more effect of $u_1$ on $y_2$ because $(|-0.5588| > |0.03338|)$ for the important mode $e^{-0.5289t}$. The response terms for both $y_1$ and $y_2$ are plotted in Figures 1 and 2 which illustrate the time scale of the interaction.

The above analysis indicates that for effective control, $y_1$ should be connected to $u_1$, and $y_2$ connected to both $u_2$ and $u_1$. The best single loop scheme is, of course, the pairs $y_1$-$u_1$ and $y_2$-$u_2$; but there will be considerable interactions between $y_2$ and $u_1$ when using this single loop control. Bristol (1966) has pointed out that negative elements in the steady state RGA indicate the possibility of inverse response for the closed loop behavior; Figures 1 and 2 illustrate that his supposition is true for $\Delta_{1,2}$ and $\Delta_{2,1}$.

It is interesting to compare the results obtained above, which are essentially of the screening variety, with control synthesis results using more complicated methods using optimal control and pole placement. Optimal control calculations for this same binary distillation column (Oakley and Edgar, 1976) have shown that the components of the optimal feedback gain matrix are heavily weighted towards the same coupling described above (reboiler control connected to both top and bottom compositions, while the reflux control needs to be connected to the overhead composition). While this result is not unexpected, optimal control calculations for such a large scale system are rather unpredictable. Pole placement using various feedback couplings for this system has been studied by Tung and Edgar (1977); they found that eigenvalues of the closed loop system could be shifted further with the preferred single loop coupling obtained by the dynamic interaction index, $(u_1 - y_2;\ u_2 - y_1)$. The steady state index coupling $(u_1 - y_1;\ u_2 - y_2)$ did not allow extensive shifting of the eigenvalues; however, time domain simulation showed that better control, requiring less control effort, was obtained with the coupling suggested by steady state analysis. This, of course, is not surprising in light of the rather unpredictable results obtained by pole placement methods. If, as suggested earlier, we allow multiple feedbacks, rather than strictly single loop, the pole placement method gives results consistent with optimal control.

## CONCLUSIONS

The results of the dynamic interaction analysis for the distillation column control system indicates that this method can be used profitably to evaluate the need for multivariable control and to indicate potential control difficulties. A similar analysis (Tung and Edgar, 1979) for a fluid catalytic cracker has also highlighted potential difficulties with single loop control. Therefore the dynamic RGA usefully augments the information provided by a static RGA. The analysis can be implemented easily with a digital computer.

## NOTATION

$a$ = interaction coefficient for a given mode
$A$ = state matrix
$Adj$ = adjoint of matrix
$B$ = control matrix
$C$ = measurement matrix
$det$ = determinant of matrix
$I$ = identity matrix
$L$ = liquid flow rate
$m$ = dimension of $y$ and $u$
$n$ = dimension of $x$
$r$ = specified value of summation index
$RGA$ = relative gain array
$s$ = Laplace transform variable
$t$ = time
$u$ = control vector
$x$ = state vector
$y$ = output vector

### Greek Letters

$\alpha_{ij}$ = element of relative gain array
$\Delta_{i,j}$ = interaction term
$\Gamma_{i,j}$ = elements of inverse of steady-state gain matrix
$\lambda_j$ = $j$th eigenvalue
$\phi_{i,j}$ = elements of transfer function matrix
$\Sigma$ = summation

### Subscripts

$i$ = row in relative gain array
$j$ = column in relative gain array
$k$ = summation index

### Superscripts

$o$ = steady state change
$\hat{}$ = new definition

## LITERATURE CITED

Bristol, E. H., "On a New Measure of Interaction for Multivariable Process Control," *IEEE Trans. Auto. Control*, AC-11, 133 (1966).
Bristol, E. H., "Recent Results on Interactions in Multivariable Process Control," AIChE 71st Annual Meeting, Miami, FL (Nov., 1978).
Gagnepain, J. P., and D. E. Seborg, "An Analysis of Process Interactions with Applications to Multiloop Control System Design," *IEC Proc. Des. Dev.*, in press (1980).
McAvoy, T. J., "Interacting Control Systems: Steady State and Dynamic Measurement of Interaction," *ISA Trans.*, **16**, No. 3, 35 (1978).
Oakley, D. R., and T. F. Edgar, "Optimal Feedback Control of a Binary Distillation Column," *Proc. JACC*, 551 (1976).
Shinskey, F. G., *Process Control Systems*, McGraw Hill, New York (1967).
Tung, L. S., and T. F. Edgar, "Dynamic Interaction Index and Its Application to Distillation Column Control," *Proc. IEEE Conference on Decision and Control*, New Orleans, LA., December, 1977.
Tung, L. S., and T. F. Edgar, "Analysis of Control-Output Interactions in Dynamic Systems," AIChE 71st Annual Meeting, Miami, FL (Nov., 1978).
Tung, L. S., and T. F. Edgar, "Development and Reduction of a Multivariable Binary Distillation Column Model with Tray Hydraulics," AIChE 86th National Meeting, Houston, TX (April, 1979).

*Manuscript received June 3, and accepted July 16, 1980.*

---

# Heat and Mass Transfer in Turbulent Flow Under Conditions of Drag Reduction

**O. T. HANNA**
**O. C. SANDALL**
**and**
**P. R. MAZET**

**University of California**
**Santa Barbara, CA**

It is well known that the addition of a small quantity of long-chain polymer to a material will cause the frictional drag to be reduced under conditions of turbulent flow (Virk, 1975). The corresponding reduction of heat or mass transfer has been the subject of a number of investigations, both theoretical and experimental.

The basic reasons for the drag reduction phenomenon are still under investigation. Up to now, mathematical descriptions and transport rate predictions have been based primarily on an eddy diffusivity model of turbulent transport. This approach will again be followed here.

Previous efforts to predict heat or mass transfer rates under conditions of drag reduction have generally involved the numerical integration of the Lyon transport equation. Various approximations have been introduced by different authors, particularly with regard to the form of the eddy diffusivity distribution under conditions of drag reduction. The more recent theoretical studies include the work of Dimant and Poreh (1976), Ghajar and Tiederman (1977), Kale (1977) and, Virk and Suraiya (1977).

Dimant and Poreh use a modified form of the Van Driest (1955) eddy diffusivity model for their numerical solutions of the energy equation. A graphical comparison of their numerical results with experimental heat transfer results corresponding to Prandtl numbers of order ten shows good agreement. The formulation of Dimant and Poreh leads to the limiting behavior $St \sim \sigma^{-3/4}$ for $\sigma \to \infty$.

Ghajar and Tiederman (1977) use an eddy diffusivity distribution due to Cess (1958), together with experimental data on frictional drag reduction, to determine their numerical evaluations of the Lyon equation for heat transfer. Their numerical results are shown graphically. It can be shown that their formulation leads to the asymptotic behavior $St \sim \sigma^{-2/3}$ for large $\sigma$.

Kale (1977) has employed a numerical evaluation of an approximate form of the Lyon equation, which is due to Reichardt (1957), for his heat transfer studies. A modified form of an eddy

0001-1541-81-3198-0694-$2.00. 